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On extension of a result of Flett for Cesáro matrices

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Abstract

In this work we prove a theorem which shows that a Cesáro matrix of order $\alpha > -1$ is a bounded operator on $\mathcal{A}_k$, defined below by (2); i.e., $(C, \alpha) \in B(\mathcal{A}_k)$.

Keywords: Absolute summability; Bounded operator; Cesáro matrix; Conservative matrix

Let $\sum a_n$ be an infinite series with partial sums (s_n), (C, α) the Cesáro matrix of order α. The concept of absolute summability of order k was introduced and studied by Flett [2]. A series $\sum a_n$ is summable $|C, \alpha|_k$, $k \geq 1$, $\alpha > -1$, if

$$\sum_{n=1}^{\infty} n^{k-1} \left|\sigma_{n-1}^{\alpha} - \sigma_n^{\alpha}\right|^k < \infty, \tag{1}$$

where σ_n^{α} denotes the nth term of the (C, α) transform of (s_n).

He also proved the following inclusion theorem. If series $\sum a_n$ is summable $|C, \alpha|_k$, it is summable $|C, \beta|_r$ for each $r \geq k \geq 1$, $\alpha > -1$, $\beta > \alpha + 1/k - 1/r$. It then follows that, if one chooses $r = k$, then a series $\sum a_n$ which is $|C, \alpha|_k$ summable is also $|C, \beta|_k$ summable for $k \geq 1$, $\beta > \alpha > -1$.

Let $\sum a_n$ be a series with partial sums s_n. Define

$$\mathcal{A}_k := \left\{ (s_n)_{n=0}^{\infty} : \sum_{n=1}^{\infty} n^{k-1} |a_n|^k < \infty; a_n = s_n - s_{n-1} \right\}. \tag{2}$$

A matrix T is said to be a bounded linear operator on $\mathcal{A}_k$, written $T \in B(\mathcal{A}_k)$, if $T : \mathcal{A}_k \to \mathcal{A}_k$.

If one sets $\alpha = 0$ in the inclusion statement involving (C, α) and (C, β), then one obtains the fact that $(C, \beta) \in B(\mathcal{A}_k)$ for each $\beta > 0$.

Let T be a sequence-to-sequence transformation transforming the sequence (s_n) into (t_n). If, whenever (s_n) converges absolutely, (t_n) converges absolutely, T is called absolutely conservative. If the absolute convergence of (s_n) implies absolute convergence of (t_n) to the same limit, T is called absolutely regular.

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Das [1] defined a matrix $T = (t_{nv})$ to be absolutely kth-power conservative for $k \geq 1$, if $T \in B(\mathcal{A}_k)$; i.e., if (s_n) is a sequence satisfying

$$\sum_{n=1}^{\infty} n^{k-1} |s_n - s_{n-1}|^k < \infty,$$

then

$$\sum_{n=1}^{\infty} n^{k-1} |t_n - t_{n-1}|^k < \infty,$$

where

$$t_n = \sum_{v=0}^{\infty} t_{nv} s_v.$$

He also showed that every conservative Hausdorff matrix $H \in B(\mathcal{A}_k)$. We know that if $\beta \geq 0$, then (C, β) is regular and if $\beta < 0$, then (C, β) is neither conservative nor regular.

In this work we extend the result of Flett to the case $\beta > -1$, thus demonstrating that being a conservative matrix is not a necessary condition for a matrix to belong to $B(\mathcal{A}_k)$.

Theorem 1. $(C, \alpha) \in B(\mathcal{A}_k)$ *for each* $\alpha > -1$.

Proof. Let σ_n^α denote the nth term of the Cesáro mean of order α of a sequence (s_n); i.e.,

$$\sigma_n^\alpha = \frac{1}{E_n^\alpha} \sum_{v=0}^{n} E_{n-v}^{\alpha-1} s_v.$$

We shall show that $\left(\sigma_n^\alpha\right) \in \mathcal{A}_k$; i.e.,

$$\sum_{n=1}^{\infty} n^{k-1} \left|\sigma_n^\alpha - \sigma_{n-1}^\alpha\right|^k < \infty. \tag{3}$$

Let τ_n^α denote the nth term of the Cesáro mean of order α ($\alpha > -1$) of the sequence (na_n); i.e.,

$$\tau_n^\alpha = \frac{1}{E_n^\alpha} \sum_{v=1}^{n} E_{n-v}^{\alpha-1} v a_v.$$

Since $\tau_n^\alpha = n(\sigma_n^\alpha - \sigma_{n-1}^\alpha)$ (see [3]), condition (3) can also be written as

$$\sum_{n=1}^{\infty} \frac{1}{n} \left|\tau_n^\alpha\right|^k < \infty. \tag{4}$$

Applying Hölder's inequality, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n} \left|\tau_n^\alpha\right|^k &= \sum_{n=1}^{\infty} \frac{1}{n} \left| \frac{1}{E_n^\alpha} \sum_{v=1}^{n} E_{n-v}^{\alpha-1} v a_v \right|^k \\ &\leq \sum_{n=1}^{\infty} \frac{1}{n(E_n^\alpha)^k} \sum_{v=1}^{n} E_{n-v}^{\alpha-1} v^k |a_v|^k \times \left\{ \sum_{v=1}^{n} E_{n-v}^{\alpha-1} \right\}^{k-1}.\end{aligned}$$

Since

$$\sum_{v=1}^{n} E_{n-v}^{\alpha-1} = E_n^{\alpha-1} + E_{n-1}^{\alpha-1} + \cdots + E_0^{\alpha-1} = E_n^\alpha,$$

we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n}\left|\tau_n^{\alpha}\right|^k &\le \sum_{n=1}^{\infty}\frac{1}{nE_n^{\alpha}}\sum_{v=1}^{n}E_{n-v}^{\alpha-1}v^k\,|a_v|^k\\ &= \sum_{v=1}^{\infty}v^k\,|a_v|^k\sum_{n=v}^{\infty}\frac{E_{n-v}^{\alpha-1}}{nE_n^{\alpha}}.\end{aligned}$$

For $\alpha > -1$, $n \ge 1$ since

$$\frac{1}{nE_n^{\alpha}} = \int_0^1 (1-x)^{\alpha}x^{n-1}\mathrm{d}x$$

and

$$(1-x)^{-\alpha} = \sum_{n=0}^{\infty}E_n^{\alpha-1}x^n,$$

we obtain

$$\begin{aligned}\sum_{n=v}^{\infty}\frac{E_{n-v}^{\alpha-1}}{nE_n^{\alpha}} &= \sum_{r=0}^{\infty}\frac{E_r^{\alpha-1}}{(v+r)E_{v+r}^{\alpha}}\\ &= \sum_{r=0}^{\infty}E_r^{\alpha-1}\int_0^1(1-x)^{\alpha}x^{v+r-1}\mathrm{d}x\\ &= \int_0^1(1-x)^{\alpha}x^{v-1}\left(\sum_{r=0}^{\infty}E_r^{\alpha-1}x^r\right)\mathrm{d}x\\ &= \int_0^1 x^{v-1}\mathrm{d}x\\ &= \frac{1}{v}.\end{aligned}$$

Then

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n}\left|\tau_n^{\alpha}\right|^k &= O(1)\sum_{v=1}^{\infty}v^k\,|a_v|^k\,\frac{1}{v}\\ &= O(1)\sum_{v=1}^{\infty}v^{k-1}\,|a_v|^k = O(1),\end{aligned}$$

since $s_n \in \mathcal{A}_k$. $\square$

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Further reading

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